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The off-critical integrable Ashkin–Teller model

Paul A Pearce†§ and Katherine A Seaton‡

† Centre for Mathematical Analysis, The Australian National University, Canberra, ACT 2601, Australia

‡ Mathematics Department, University of Melbourne, Parkville, Vic. 3052, Australia

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Abstract. The exact solution of an Ashkin-Teller model, with layer and spatial anisotropy and $Z_2 \times Z_2$ symmetry, is presented. The model is related to the eight-vertex model by a sublattice or orbifold duality and it has the same free energy and magnetisation. The exact solution manifold is a three-dimensional submanifold of the full six-dimensional thermodynamic space. It admits a Z_3 symmetry relating three regimes each exhibiting different partial order. In the spatially isotropic case, these regimes are separated by three lines of continuously varying critical behaviour which coalesce at the four-state Potts critical point. The magnetisations and polarisation are derived using corner transfer matrices giving rise to the appearance of chiral and Virasoro characters. The associated critical exponents confirm the known values. In particular, we obtain the value $x_\sigma = \frac{1}{8}$ for the magnetic scaling dimension and identify the next-to-leading thermal field.

1. Introduction

The Ashkin-Teller model (Ashkin and Teller 1943) was first proposed as a generalisation of the Ising model to describe a four-component alloy. The general model has a $Z_2 \times Z_2$ symmetry and consists (Fan 1972a) of two anisotropic square lattice Ising models coupled by four-spin interactions. The phase diagram of this model has been much studied (Fan and Wu 1970, Fan 1972b, Wegner 1972, Wu and Lin 1974, Knops 1975, Ditzian *et al* 1980, Pfister 1982) particularly in connection with the duality properties of the model. For spatially isotropic interactions, the model exhibits three self-dual critical lines which intersect at the four-state Potts critical point. Each of these lines exhibits continuously varying critical behaviour described by a conformal theory with central charge $c = 1$.

The critical behaviour of the Ashkin-Teller model on a self-dual line can be understood (Kadanoff 1979, Kadanoff and Brown 1979, Knops 1980, den Nijs 1981) by using the renormalisation group to map the model onto the Gaussian model. The continuously varying scaling dimensions are given by

$$x_{m,n} = \frac{m^2}{4(2-y)} + n^2(2-y) \tag{1.1a}$$

where m and n are arbitrary integers and the renormalisation-group exponent y varies along the critical line over the range $0 < y < \frac{4}{3}$. These dimensions are augmented by a

§ Permanent address: Mathematics Department, University of Melbourne, Parkville, Vic 3052, Australia.

sector of fixed scaling dimensions (Yang 1987, Saleur 1987). The list of relevant primary operators in this sector is

$$x_0 = 0 \text{ (identity)} \qquad x_\sigma = \tfrac{1}{8} \text{ (magnetisation)}$$
$$x'_\sigma = \tfrac{9}{8} \text{ (second magnetisation)} \qquad x = 2 \text{ (marginal operator)}. \tag{1.1b}$$

The Ashkin–Teller model can be readily solved on the self-dual line. In this paper, we solve the Ashkin–Teller model on a more general non-critical manifold exhibiting three distinct forms of partial order. This manifold extends off the self-dual critical line in the direction of the next-to-leading scaling field p with scaling dimension

$$x_p = x_{0,1} = 2 - y. \tag{1.1c}$$

Our results for the free energy and magnetisation exponents confirm the known values. In particular, we find the critical exponents

$$2 - \bar{\alpha} = 2/y \qquad \bar{\beta}_\sigma = 1/8y \tag{1.2a}$$

where overbars indicate exponents obtained by approaching the critical line along the exact solution manifold, i.e. using p as the deviation-from-criticality variable. We conclude that the value of the magnetic scaling dimension is

$$x_\sigma = 2\Delta_\sigma = \frac{2\bar{\beta}_\sigma}{2 - \bar{\alpha}} = \tfrac{1}{8}. \tag{1.2b}$$

The layout of the paper is as follows. In section 2 we define the general anisotropic Ashkin–Teller model, describe its symmetries and phase diagram and derive an elliptic parametrisation on the exact solution manifold. In section 3, we use corner transfer matrices to calculate the magnetisations and polarisation on the exact solution manifold. We also write the traces of the diagonal blocks of the corner transfer matrices as products of two $c = \frac{1}{2}$ Virasoro characters. The critical behaviour and its connection with scaling theory is discussed in section 4.

2. The model

2.1. *The phase diagram and exact solution manifold*

The anisotropic Ashkin–Teller model on a square lattice is an interaction-round-a-face or IRF model (Baxter 1982). A four-state compound spin $\sigma_i = (s_i, t_i)$, where $s_i, t_i = \pm 1$ are Ising spins, is associated with each site i of the lattice. The Hamiltonian H is given by

$$\beta H = -\sum_{\text{horiz}} (K s_i s_j + L t_i t_j + M s_i s_j t_i t_j) - \sum_{\text{vert}} (K' s_i s_j + L' t_i t_j + M' s_i s_j t_i t_j) \tag{2.1}$$

where $\beta = 1/k_B T$ is the inverse temperature and the sums are over the horizontal and vertical bonds of the lattice, respectively. We will assume that the two- and four-spin interactions are ferromagnetic, i.e. K, K', L, L', M and $M' \geq 0$. The Hamiltonian (2.1) is invariant under two independent spin reversal symmetries:

$$\mathcal{R}_1: \qquad s_i \leftrightarrow -s_i \qquad \mathcal{R}_1^2 = I \tag{2.2a}$$

$$\mathcal{R}_2: \qquad t_i \leftrightarrow -t_i \qquad \mathcal{R}_2^2 = I. \tag{2.2b}$$

These symmetries, which are spontaneously broken in the ordered phases, are the origin of the generic $Z_2 \times Z_2$ symmetry of the model. The total spin reversal operator is $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$. The Ashkin-Teller interactions allow for two types of anisotropy: spatial and layer anisotropy. We say the model is spatially isotropic if $K = K'$, $L = L'$, $M = M'$ and layer isotropic if $K = L$, $K' = L'$. In the case of layer isotropy, the model has the dihedral group D_4 as its symmetry group. In the completely symmetric case ($K = L = M$, $K' = L' = M'$), the Ashkin-Teller model reduces to the spatially anisotropic four-state Potts model with the full symmetry of the symmetric group S_4.

Under the *A-D-E* classification (Pasquier 1987a, b, c, Ginsparg 1988), the critical four-state Potts and critical Ashkin-Teller models are associated with the affine Lie algebra $D_4^{(1)}$. The critical *A-D-E* models are solvable. Furthermore, it is known that each critical model in the *A* or *D* series admits (Kuniba and Yajima 1987) an off-critical integrable extension involving elliptic functions. The Ashkin-Teller model we consider here is the most general off-critical integrable extension to the critical $D_4^{(1)}$ model. To cast the Ashkin-Teller model in the $D_4^{(1)}$ representation, we place non-interacting spins on sites added to the centre of each face of the lattice. The state of the spins on these sites is fixed to a new fifth state $\sigma_i = 0 = (0, 0)$. The five allowed spin states can then be described by the adjacency diagram of figure 1. This graph is the Dynkin diagram of the affine Lie algebra $D_4^{(1)}$. The new square lattice, containing the additional sites, is rotated at 45° to the original lattice as shown in figure 2. Each of its faces contains one bond of the original lattice, either horizontal or vertical, and is coloured black or white accordingly in a checkerboard fashion.

The Ashkin-Teller partition function is given by

$$Z = \sum_{\text{spins}} \exp(-\beta H) = \sum_{\text{spins}} \prod_{(i,j,k,l)} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2.3}$$

where the sums are over all spins σ_i, the product is over all square faces (i, j, k, l) of the new lattice, and the Boltzmann weights of black and white faces are given by

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = W_1(\sigma_i, \sigma_k) \tag{2.4a}$$

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = W_2(\sigma_j, \sigma_l) \tag{2.4b}$$

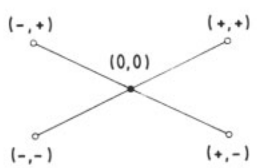

Figure 1. The Dynkin diagram of the Lie algebra $D_4^{(1)}$. This graph is the adjacency diagram of the Ashkin-Teller and four-state Potts models.

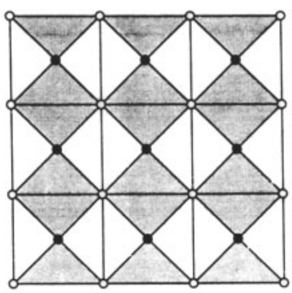

Figure 2. The anisotropic Ashkin-Teller model (2.1) is defined on the square lattice indicated by open circles. A rotated square lattice, consisting of open and solid circles, is obtained by adding fixed spins at the centre of each square face of this lattice. Black and white faces contain the horizontal and vertical interactions, respectively. On the new lattice the conditions on the states of adjacent spins is specified by the Dynkin diagram of figure 1.

with edge weights

$$W_1(\sigma_i, \sigma_j) = \exp(Ks_is_j + Lt_it_j + Ms_is_jt_it_j) \tag{2.5a}$$

$$W_2(\sigma_i, \sigma_j) = \exp(K's_is_j + L't_it_j + M's_is_jt_it_j). \tag{2.5b}$$

The partition function (2.3) admits symmetries in addition to the spin reversal symmetries $\mathcal{R}_1$ and $\mathcal{R}_2$. Specifically, it remains invariant if the Hamiltonian H is replaced by the Hamiltonians $\mathcal{S}H$ or $\mathcal{T}H$ obtained under the action of the transformations:

$$\mathcal{S}: \qquad s_i \leftrightarrow t_i \qquad\qquad \mathcal{S}^2 = I \tag{2.6a}$$

$$\mathcal{T}: \qquad s_i \to t_i \quad t_i \to s_it_i \qquad\qquad \mathcal{T}^3 = I. \tag{2.6b}$$

The Z_2 symmetry $\mathcal{S}$, which effectively interchanges the pairs of interactions K with L and K' with L', is in fact a duality transformation. Two other duality transformations, which interchange the other pairs, can be obtained by conjugation with $\mathcal{T}$. The Z_3 symmetry $\mathcal{T}$ acts to cyclically permute the interactions (K, L, M) and (K', L', M'). It follows that the magnetisations $\langle s_1\rangle_H$, $\langle t_1\rangle_H$ and the polarisation $\langle s_1t_1\rangle_H$ are related by

$$\langle t_1\rangle_H = \langle s_1\rangle_{\mathcal{S}H} \tag{2.7a}$$

$$\langle s_1t_1\rangle_H = \langle s_1\rangle_{\mathcal{T}H} \tag{2.7b}$$

where

$$\langle s_1\rangle_H = Z^{-1} \sum_{\text{spins}} s_1 \exp(-\beta H) \tag{2.7c}$$

etc, and we will usually drop the suffix H. Clearly, in the spatially isotropic case it is only necessary to consider a fundamental regime such as $0 \leq M \leq L \leq K$. The results for the other regimes can then be obtained by application of the transformations $\mathcal{S}$ and $\mathcal{T}$.

The exact solution manifold of the anisotropic Ashkin–Teller model is a three-dimensional manifold in the thermodynamic space spanned by the six interactions K, L, M, K', L', M'. It is specified by the edge weights (Fendley and Ginsparg 1989)

$$\begin{aligned}
W_1(\sigma, \sigma) &= (c+a)r/\sqrt{2} & W_2(\sigma, \sigma) &= (c+b)/\sqrt{2}r \\
W_1(\sigma, \mathcal{R}\sigma) &= (c-a)r/\sqrt{2} & W_2(\sigma, \mathcal{R}\sigma) &= (c-b)/\sqrt{2}r \\
W_1(\sigma, \mathcal{R}_1\sigma) &= (b+d)r/\sqrt{2} & W_2(\sigma, \mathcal{R}_1\sigma) &= (a+d)/\sqrt{2}r \\
W_1(\sigma, \mathcal{R}_2\sigma) &= (b-d)r/\sqrt{2} & W_2(\sigma, \mathcal{R}_2\sigma) &= (a-d)/\sqrt{2}r
\end{aligned} \tag{2.8}$$

where a, b, c, d are arbitrary subject only to the overall normalisation

$$(c^2-a^2)(b^2-d^2)(c^2-b^2)(a^2-d^2) = 16. \tag{2.9}$$

The gauge factor r is also arbitrary, but cancels out of products of the weights if there are equal numbers of black and white faces. This parametrisation arises by performing a duality transformation (Wegner 1972) on one sublattice of the eight-vertex model with vertex weights a, b, c, d. This is equivalent to the orbifold duality of Fendley and Ginsparg (1989). The general anisotropic Ashkin–Teller model (2.1) is in fact related (Wu 1977) by duality in this way to a staggered eight-vertex model. This staggered model also has six interaction parameters occurring in pairs; four two-spin and two four-spin couplings. Requiring this eight-vertex model to be homogeneous, and hence solvable, gives rise to three constraints among the six interactions. After some algebra, it follows from these constraints, or equivalently the constraints implied by (2.8), that the interactions of the corresponding solvable Ashkin-Teller model must satisfy:

$$\Delta_1 = \frac{\sinh 2K}{\sinh 2L} = \frac{\sinh 2K'}{\sinh 2L'} \tag{2.10a}$$

$$\Delta_2 = \frac{\sinh 2M}{\sinh 2L} = \frac{\sinh 2M'}{\sinh 2L'} \tag{2.10b}$$

$$\exp(2M + 2M') \frac{\sinh(K+L)\sinh(K'+L')}{\cosh(K-L)\cosh(K'-L')} = 1. \tag{2.10c}$$

In the spatially isotropic case, these reduce to the single constraint

$$\exp(2K + 2L + 2M) = \exp(2K) + \exp(2L) + \exp(2M) \tag{2.11}$$

which describes a two-dimensional surface in the three-dimensional thermodynamic space spanned by K, L, M. This exact solution surface is shown in figure 3. The complete phase diagram of the spatially isotropic Ashkin-Teller model, as determined by Wu and Lin (1974), is shown in figure 4.

The Ashkin-Teller model has previously been solved (see for example Baxter 1982) for layer isotropic interactions ($K = L$, $K' = L'$). The exact solution manifold in this case is given by a specialisation of the constraints (2.10). For interactions that are

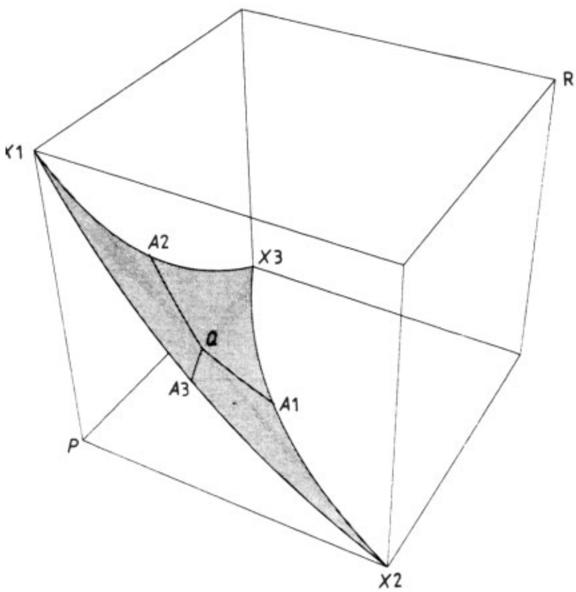

Figure 3. The exact solution manifold $X1X2X3$ of the spatially isotropic Ashkin-Teller model ($K = K'$, $L = L'$, $M = M'$) given by $\exp(2K+2L+2M) = \exp(2K) + \exp(2L) + \exp(2M)$. For convenience in plotting, the coordinates are $(x, y, z) = (\tanh K, \tanh L, \tanh M)$ with $K, L, M \geq 0$. The line PQR, given by $K = L = M$, is the four-state Potts line where the points R and P correspond to zero and infinite temperature, respectively. The Ashkin-Teller model has a Z_3 symmetry about the line PQR generated by permuting the interactions K, L and M. The three lines of continuously varying critical behaviour $A1Q$, $A2Q$ and $A3Q$ coalesce at the four-state Potts critical point Q.

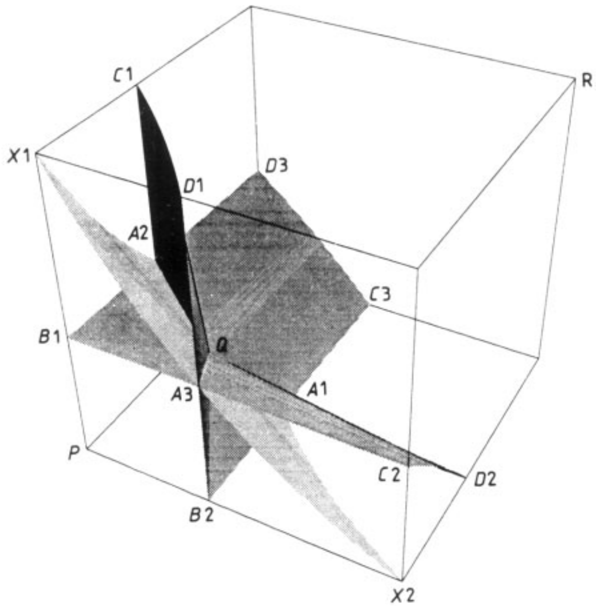

Figure 4. The phase diagram of the spatially isotropic Ashkin–Teller model plotted using the same coordinates as in figure 3. The topology of the phase diagram is given by Wu and Lin (1974). The thermodynamic space is divided into five phases: one ferromagnetic, one disordered and three partially ordered. These phases are separated by six critical surfaces which intersect along the three lines of continuously varying critical behaviour. The six surfaces, given by $B1A2QA3$, $C1D1A3QA2$ and their images under the Z_3 symmetry, should exhibit Ising critical exponents reflecting Z_2 symmetry breaking. The exact solution manifold $X1X2X3$ only intersects the partially ordered regimes. The model reduces to Ising models with known critical lines on the six faces of the cube.

both spatially and layer isotropic, the surface (2.11) reduces to the known self-dual line (Fan 1972a)

$$\sinh 2K = \exp(-2M) \qquad K = L. \tag{2.12}$$

If $M < K$, this is a critical line with continuously varying critical behaviour. On the other hand, if $M > K$, this line lies entirely within a partially ordered region with $\langle s_1\rangle = \langle t_1\rangle = 0$ and $\langle s_1 t_1\rangle > 0$ as shown by Seaton and Pearce (1989). If $M = 0$, the model decouples into two independent Ising models with critical lines given by

$$\sinh 2K = 1 \qquad L \text{ arbitrary} \tag{2.13a}$$

$$\sinh 2L = 1 \qquad K \text{ arbitrary.} \tag{2.13b}$$

Similarly, in the limit $M \to \infty$, we must have $s_i s_j = t_i t_j$, so the model reduces to an Ising model with interaction $K + L$ and a critical line given by

$$\sinh 2(K + L) = 1. \tag{2.13c}$$

Critical lines equivalent to (2.12) and (2.13) are obtained by permuting K, L and M. The three lines of continuously varying critical behaviour so obtained intersect at the four-state Potts critical point

$$K = L = M = \tfrac{1}{4}\log 3. \tag{2.14}$$

The other critical lines equivalent to (2.13) join to define the boundaries of critical surfaces as shown in figure 4. The phase diagram is divided into five regions: three partially ordered (one of $\langle s_1\rangle$, $\langle t_1\rangle$ or $\langle s_1 t_1\rangle$ positive), one disordered ($\langle s_1\rangle = \langle t_1\rangle = \langle s_1 t_1\rangle = 0$) and one with complete ferromagnetic order ($\langle s_1\rangle$, $\langle t_1\rangle$, $\langle s_1 t_1\rangle > 0$). The six critical surfaces separating these phases should exhibit Ising critical exponents reflecting the fact that one of the two Z_2 symmetries is spontaneously broken as each surface is crossed. Note that the layer isotropic phase diagram (Baxter 1982), which exhibits only one partially ordered region, is a diagonal cross-section of the three-dimensional phase diagram.

2.2. *Elliptic parametrisation*

The exact solution manifold of the anisotropic Ashkin–Teller model can be conveniently parametrised in terms of elliptic functions. One way to do this is to start with the parametrisation of the eight-vertex weights

$$\begin{aligned} a-b&=\rho\frac{\theta_1((\lambda/2)-u)}{\theta_1(\lambda/2)} & a+b&=\rho\frac{\theta_2((\lambda/2)-u)}{\theta_2(\lambda/2)} \\ c-d&=\rho\frac{\theta_4((\lambda/2)-u)}{\theta_4(\lambda/2)} & c+d&=\rho\frac{\theta_3((\lambda/2)-u)}{\theta_3(\lambda/2)}. \end{aligned} \tag{2.15}$$

Here u is a spectral parameter, λ is the crossing parameter and θ_1, θ_2, θ_3, θ_4 are standard elliptic functions of nome p, with $|p|<1$, defined (Gradshteyn and Rhyzik 1965) by

$$\theta_1(u)=\theta_1(u,p)=2p^{1/4}\sin u\prod_{n=1}^{\infty}(1-2p^{2n}\cos 2u+p^{4n})(1-p^{2n}) \tag{2.16a}$$

$$\theta_2(u)=\theta_2(u,p)=2p^{1/4}\cos u\prod_{n=1}^{\infty}(1+2p^{2n}\cos 2u+p^{4n})(1-p^{2n}) \tag{2.16b}$$

$$\theta_3(u)=\theta_3(u,p)=\prod_{n=1}^{\infty}(1+2p^{2n-1}\cos 2u+p^{4n-2})(1-p^{2n}) \tag{2.16c}$$

$$\theta_4(u)=\theta_4(u,p)=\prod_{n=1}^{\infty}(1-2p^{2n-1}\cos 2u+p^{4n-2})(1-p^{2n}). \tag{2.16d}$$

The prefactor ρ is fixed by the normalisation (2.9). Application of elliptic function identities leads to the following expressions for the parametrised edge weights:

$$\begin{aligned} W_1(\sigma,\sigma)&=\sqrt{2}\,\rho r\frac{\theta_1(\lambda-(u/2))\theta_2(u/2)\theta_3(u/2)\theta_4(u/2)}{\theta_1(\lambda)\theta_2(0)\theta_3(0)\theta_4(0)}=\frac{\rho r}{\sqrt{2}}\frac{\theta_1(u)\theta_1(\lambda-(u/2))}{\theta_1(\lambda)\theta_1(u/2)} \\ W_1(\sigma,\mathscr{R}\sigma)&=\sqrt{2}\,\rho r\frac{\theta_1(u/2)\theta_2(\lambda-(u/2))\theta_3(u/2)\theta_4(u/2)}{\theta_1(\lambda)\theta_2(0)\theta_3(0)\theta_4(0)}=\frac{\rho r}{\sqrt{2}}\frac{\theta_1(u)\theta_2(\lambda-(u/2))}{\theta_1(\lambda)\theta_2(u/2)} \\ W_1(\sigma,\mathscr{R}_1\sigma)&=\sqrt{2}\,\rho r\frac{\theta_1(u/2)\theta_2(u/2)\theta_3(u/2)\theta_4(\lambda-(u/2))}{\theta_1(\lambda)\theta_2(0)\theta_3(0)\theta_4(0)}=\frac{\rho r}{\sqrt{2}}\frac{\theta_1(u)\theta_4(\lambda-(u/2))}{\theta_1(\lambda)\theta_4(u/2)} \\ W_1(\sigma,\mathscr{R}_2\sigma)&=\sqrt{2}\,\rho r\frac{\theta_1(u/2)\theta_2(u/2)\theta_3(\lambda-(u/2))\theta_4(u/2)}{\theta_1(\lambda)\theta_2(0)\theta_3(0)\theta_4(0)}=\frac{\rho r}{\sqrt{2}}\frac{\theta_1(u)\theta_3(\lambda-(u/2))}{\theta_1(\lambda)\theta_3(u/2)} \end{aligned} \tag{2.17a}$$

with similar formulas for the edge weights $W_2(\sigma, \sigma')$ given by the crossing symmetry

$$W_2(\sigma, \sigma'|u, r) = W_1(\sigma, \sigma'|\lambda - u, 1/r). \tag{2.17b}$$

These weights were observed in passing to satisfy the Yang-Baxter or star-triangle equations by Kashiwara and Miwa (1986) in their study of N-state models with broken Z_N symmetry for the special case $N = 4$. The explicit parametrisation of the interactions takes the form

$$\exp(4K) = \frac{\theta_1(\lambda - (u/2))\theta_2(u/2)\theta_3(u/2)\theta_4(\lambda - (u/2))}{\theta_1(u/2)\theta_2(\lambda - (u/2))\theta_3(\lambda - (u/2))\theta_4(u/2)} \tag{2.18a}$$

$$\exp(4L) = \frac{\theta_1(\lambda - (u/2))\theta_2(u/2)\theta_3(\lambda - (u/2))\theta_4(u/2)}{\theta_1(u/2)\theta_2(\lambda - (u/2))\theta_3(u/2)\theta_4(\lambda - (u/2))} \tag{2.18b}$$

$$\exp(4M) = \frac{\theta_1(\lambda - (u/2))\theta_2(\lambda - (u/2))\theta_3(u/2)\theta_4(u/2)}{\theta_1(u/2)\theta_2(u/2)\theta_3(\lambda - (u/2))\theta_4(\lambda - (u/2))}. \tag{2.18c}$$

Again the other interactions K', L', M' are obtained by replacing the spectral parameter u with $\lambda - u$ in these expressions. Also, applying elliptic function identities to the invariants (2.10a) and (2.10b), we obtain

$$\Delta_1 = \frac{\theta_3(0)\theta_4(\lambda)}{\theta_3(\lambda)\theta_4(0)} \tag{2.19a}$$

$$\Delta_2 = \frac{\theta_2(\lambda)\theta_3(0)}{\theta_2(0)\theta_3(\lambda)}. \tag{2.19b}$$

In the following we will restrict the elliptic parameters to the regime

$$0 < u < \lambda < 2\pi/3 \qquad 0 < p < 1. \tag{2.20}$$

With this restriction the elliptic parametrisation covers one sixth of the exact solution manifold. Clearly, the spatially isotropic case ($K = K'$, $L = L'$, $M = M'$) is given by $u = \lambda/2$ and the spectral parameter u is a measure of spatial anisotropy. Indeed, the crossing transformation $u \to \lambda - u$ implements a rotation through 90°. The special point $\lambda = 0$ corresponds to the four-state Potts critical point (2.14). Similarly, when $\lambda = \pi/2$, it is seen that $M = 0$ so the Ashkin-Teller model decouples into two independent Ising models. At $\lambda = \pi/4$, the model reduces to the four-state model with broken Z_4 symmetry solved by Kashiwara and Miwa (1986) and Jimbo *et al* (1986). From (2.18) it is also seen that the duality transformation ($K \leftrightarrow L$, $K' \leftrightarrow L'$) is implemented by changing the sign of the elliptic nome ($p \leftrightarrow -p$) which has the effect of interchanging θ_3 and θ_4. The self-dual line thus occurs for $p = 0$.

The parametrised Ashkin-Teller weights (2.17) satisfy the star-triangle or Yang-Baxter equations as a consequence of duality (Fendley and Ginsparg 1989). The model can therefore be solved using commuting transfer matrix methods. Of course, the parametrised weights (2.17) with u replaced by iu and λ replaced with iλ also satisfy the star-triangle equations. This is not a new solution, however, since it is simply related to the original solution through the conjugate modulus transformation

$$\begin{aligned}
\theta_1(u, e^{-\varepsilon}) &= -i(\pi/\varepsilon)^{1/2} \exp(-u^2/\varepsilon)\theta_1(\pi i u/\varepsilon, \exp(-\pi^2/\varepsilon)) \\
\theta_2(u, e^{-\varepsilon}) &= (\pi/\varepsilon)^{1/2} \exp(-u^2/\varepsilon)\theta_4(\pi i u/\varepsilon, \exp(-\pi^2/\varepsilon)) \\
\theta_3(u, e^{-\varepsilon}) &= (\pi/\varepsilon)^{1/2} \exp(-u^2/\varepsilon)\theta_3(\pi i u/\varepsilon, \exp(-\pi^2/\varepsilon)) \\
\theta_4(u, e^{-\varepsilon}) &= (\pi/\varepsilon)^{1/2} \exp(-u^2/\varepsilon)\theta_2(\pi i u/\varepsilon, \exp(-\pi^2/\varepsilon))
\end{aligned} \tag{2.21}$$

where $p=\exp(-\varepsilon)$. Notice that the conjugate modulus transformation acts to interchange θ_2 and θ_4. By successive applications of the duality and conjugate modulus transformations, it is possible to cover the entire exact solution manifold either in terms of the original variables u, λ, $\exp(-\varepsilon)$ or in terms of the conjugate modulus variables $\pi i u/\varepsilon$, $\pi i\lambda/\varepsilon$, $\exp(-\pi^2/\varepsilon)$. Finally, we observe that, in terms of these transformations, the Z_3 symmetry $\mathcal{T}$ can be realised as a cyclic permutation of θ_2, θ_3 and θ_4.

3. Calculation of the order parameters

In this section we use corner transfer matrices (CTM) to calculate the magnetisations and polarisation using the elliptic parametrisation in the fundamental regime (2.20). The results are extended to the rest of the exact solution manifold (2.10) by applying the symmetries (2.7).

Let us define normalised corner transfer matrices (CTM) A, B, C, D in the usual way (Baxter 1981, 1982) so that A corresponds to the lower-right corner of the lattice. The elements of $A(u)$ are given by

$$A(u)_{\sigma,\sigma'}=\alpha^{-1}\delta(\sigma_1,\sigma_1')\sum_{\text{spins}}\prod_{(i,j,k,l)}W(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \tag{3.1}$$

where the product is over all $m(m+1)/2$ faces of the corner, the sum is over all interior spins and $\sigma=(\sigma_1,\ldots,\sigma_m)$ and $\sigma'=(\sigma_1',\ldots,\sigma_m')$ are the m spins along the left and upper edges of the corner respectively. The corner spin is $\sigma_1=\sigma_1'=(s_1,t_1)$. The spins on the perimeter are fixed to the groundstate values given by $\sigma_i=(0,0)$ on one sublattice, which we call the even sublattice, and $\sigma_i=(+,+)$ on the odd sublattice. The normalisation constant α is chosen so that the groundstate element of $A(u)$ is unity. Similarly, we define corner transfer matrices $B(u)$, $C(u)$, $D(u)$ corresponding respectively to the upper-right, upper-left and lower-left corners of the lattice. These are normalised so that the groundstate elements of $B(u)$, $C(u)$ and $D(u)$ are unity. Fixing the gauge $r=1$, it follows from the crossing symmetry (2.17*b*) that

$$C(u)=A(u)\qquad B(u)=D(u)=A(\lambda-u). \tag{3.2}$$

We will assume that the spin σ_1 on the corner site is not a fixed spin, i.e. the corner site is on the odd sublattice. The two magnetisations of the Ashkin–Teller model can then be written as

$$\langle s_1\rangle=\frac{\operatorname{Tr} SABCD}{\operatorname{Tr} ABCD}\qquad \langle t_1\rangle=\frac{\operatorname{Tr} TABCD}{\operatorname{Tr} ABCD} \tag{3.3a}$$

where the elements of the matrices S and T are

$$S_{\sigma,\sigma'}=s_1\prod_{i=1}^{m}\delta(\sigma_i,\sigma_i')\qquad T_{\sigma,\sigma'}=t_1\prod_{i=1}^{m}\delta(\sigma_i,\sigma_i'). \tag{3.3b}$$

Similarly, the polarisation is

$$\langle s_1 t_1\rangle=\frac{\operatorname{Tr} STABCD}{\operatorname{Tr} ABCD}. \tag{3.3c}$$

Clearly, these one-point functions are independent of the gauge r and the normalisations of A, B, C and D.

In the limit of large m, the corner transfer matrices can be diagonalised (Baxter 1982). In particular, the eigenvalues of $A(u)$ take the simple form

$$A_d(u)_{\sigma,\sigma} = m_\sigma \exp(-\alpha_\sigma u) \tag{3.4}$$

where $A_d(u)$ denotes the diagonal form of $A(u)$ and the values of the constants m_σ and α_σ can be determined from special limiting cases. The face weights satisfy the initial condition

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l|1, 0) = (\sqrt{2})^{\pm 1}\rho\delta(\sigma_i, \sigma_k) \tag{3.5}$$

where the plus and minus signs are taken for black and white faces respectively, as in the groundstate. It follows that

$$A(0)_{\sigma,\sigma'} = \prod_{i=1}^{m} \delta(\sigma_i, \sigma_i') \tag{3.6}$$

and so comparison with (3.4) gives

$$m_\sigma = 1. \tag{3.7}$$

To determine the remaining constant α_σ, we first define

$$q = \exp(-\pi^2/\varepsilon) \qquad x = \exp(-\pi\lambda/\varepsilon) \qquad w = \exp(-\pi u/\varepsilon) \tag{3.8}$$

with $p = \exp(-\varepsilon)$ and perform a conjugate modulus transformation as in (2.21). Defining the elliptic function

$$E(w, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n^2-n)/2} w^n = \prod_{n=1}^{\infty} (1 - q^{n-1}w)(1 - q^n w^{-1})(1 - q^n) \tag{3.9}$$

we obtain

$$\begin{aligned}
W_1(\sigma, \sigma) &= \frac{\rho' r}{\sqrt{2}} \frac{E(w^2, q^2)E(x^2w^{-1}, q^2)}{E(x^2, q^2)E(w, q^2)} \\
W_1(\sigma, \mathscr{R}\sigma) &= \frac{\rho' r}{\sqrt{2}} xw^{-1} \frac{E(w^2, q^2)E(qx^2w^{-1}, q^2)}{E(x^2, q^2)E(qw, q^2)} \\
W_1(\sigma, \mathscr{R}_1\sigma) &= \frac{\rho' r}{\sqrt{2}} \frac{E(w^2, q^2)E(-x^2w^{-1}, q^2)}{E(x^2, q^2)E(-w, q^2)} \\
W_1(\sigma, \mathscr{R}_2\sigma) &= \frac{\rho' r}{\sqrt{2}} xw^{-1} \frac{E(w^2, q^2)E(-qx^2w^{-1}, q^2)}{E(x^2, q^2)E(-qw, q^2)}
\end{aligned} \tag{3.10}$$

where

$$\rho' = \rho \exp[u(\lambda - u)/\varepsilon]. \tag{3.11}$$

Again the edge weights $W_2(\sigma_i, \sigma_j)$ can be obtained from (3.10) by using the crossing symmetry, i.e. by replacing w with xw^{-1} and r with r^{-1}.

Take $W_1(\sigma, \sigma')$ and $W_2(\sigma, \sigma')$ to be the entries of matrices W_1 and W_2, with rows and columns indexed by $(+,+)$, $(-,-)$, $(-,+)$, $(+,-)$. Recalling that we have set

$r=1$ we find, in the limit $x \to 0$, $w \sim 1$, that

$$W_1 \sim \frac{\rho}{\sqrt{2}} \begin{pmatrix} 1+w & 0 & 1-w & 0 \\ 0 & 1+w & 0 & 1-w \\ 1-w & 0 & 1+w & 0 \\ 0 & 1-w & 0 & 1+w \end{pmatrix} \tag{3.12a}$$

$$W_2 \sim \frac{\rho}{\sqrt{2}} \begin{pmatrix} 1 & w & 1 & w \\ w & 1 & w & 1 \\ 1 & w & 1 & w \\ w & 1 & w & 1 \end{pmatrix}. \tag{3.12b}$$

The matrix W_1 is easily diagonalised as $W_1 = P^{-1}DP$ where

$$D = \sqrt{2}\rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \tag{3.13a}$$

and the orthogonal matrix P is given by

$$P = P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{3.13b}$$

We therefore conclude that in this limit

$$W(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \sim (\sqrt{2})^{\pm 1} \rho w^{H(\sigma_l, \sigma_i, \sigma_j)} \delta(\sigma_i, \sigma_k) \tag{3.14a}$$

where

$$H(0, \sigma, 0) = (1 - st)/2 \tag{3.14b}$$

$$H(\sigma, 0, \sigma') = (1 - tt')/2. \tag{3.14c}$$

Here $0 = (0, 0)$ is the fixed spin on the even sublattice and $\sigma = (s, t)$, $\sigma' = (s', t')$ are the four-state compound Ising spins on the odd sublattice. Hence, if the boundary conditions are such that σ_{m+1} and σ_{m+2} are the values of the spins on the two sublattices at the boundary, then in this limit

$$A_d(u)_{\sigma,\sigma} = w^{n_\sigma} \tag{3.15a}$$

where the integer n_σ is given by

$$n_\sigma = \sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2}). \tag{3.15b}$$

From (3.4) and (3.8), we see that the remaining undetermined coefficients in the eigenvalues are given by

$$\alpha_\sigma = n_\sigma \pi / \varepsilon \tag{3.16}$$

and so, from (3.4), the eigenvalues of $A(u)$ are given by (3.15) throughout the regime (2.20).

Let us write

$$2n_\sigma = (1-\mu_1) + 2(1-\mu_2) + 3(1-\mu_3) + 4(1-\mu_4) + \dots \tag{3.17}$$

where the Ising spins μ_{i-1}, μ_i, s_i and t_i with i odd are related by

$$\begin{aligned} \mu_i &= t_i t_{i+2} \qquad & t_i &= \mu_i \mu_{i+2} \mu_{i+4} \dots \\ \mu_{i-1} &= s_i t_i \qquad & s_i &= \mu_{i-1} \mu_i \mu_{i+2} \mu_{i+4} \dots . \end{aligned} \tag{3.18}$$

It is clear that the even sublattice with fixed spins has dropped out as it should. Moreover, the diagonal corner transfer matrices can be written as simple direct products

$$\begin{aligned} &A_d(u) = B_d(\lambda - u) = C_d(u) = D_d(\lambda - u) \\ &A_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & w^5 \end{pmatrix} \otimes \dots \\ &T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \dots \\ &S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \dots \end{aligned} \tag{3.19}$$

with $w = \exp(-\pi u/\varepsilon)$. Putting these expressions into (3.3) and taking the thermodynamic limit, we therefore obtain

$$\langle t_1 \rangle = \prod_{n=1}^{\infty} \left(\frac{1 - x^{4n-2}}{1 + x^{4n-2}} \right) \qquad \langle s_1 \rangle = \langle s_1 t_1 \rangle = 0 \tag{3.20}$$

with $x = \exp(-\pi\lambda/\varepsilon)$. This establishes that the fundamental regime, and hence the entire exact solution manifold apart from the critical manifold, is partially ordered with one of the three order parameters being non-zero. In the limit $\lambda = \pi/2$, the Ashkin–Teller model decouples into two independent Ising models and the magnetisation reduces to the Ising result.

To examine the critical behaviour we perform a conjugate modulus transformation back to the variables p and λ. This yields the result

$$\langle t_1 \rangle = \sqrt{2}\, p^{\pi/16\lambda} \prod_{n=1}^{\infty} \left(\frac{1 + p^{n\pi/\lambda}}{1 + p^{(2n-1)\pi/2\lambda}} \right). \tag{3.21}$$

Taking p to be the deviation-from-criticality variable, we see that as $p \to 0$ the magnetisation has the power-law behaviour

$$\langle t_1 \rangle \sim p^{\bar\beta_\sigma} \qquad \bar\beta_\sigma = \pi/16\lambda. \tag{3.22}$$

The result in (3.20) for the magnetisation of the Ashkin–Teller model is identical to the formula for the magnetisation of the eight-vertex model (Baxter 1982). This is a consequence of the sublattice duality relation between the models. Indeed, the results for the magnetisations in (3.20) but not the polarisation could be inferred from duality arguments alone using the known results for the eight-vertex model. The derivation of these results using corner transfer matrices allows us to make a connection between so-called 'one-dimensional partition sums' and Virasoro characters. It has been observed (Date *et al* 1987, Kuniba and Yajima 1988, Cardy 1989, Saleur and Bauer 1989) that the spectrum of corner transfer matrices, with the centre spin set to a particular value, yields Virasoro characters. To fix the centre spin $\sigma_1 = (s_1, t_1)$ we insert a projection matrix into the trace over corner transfer matrices and define the 'one-dimensional partition sums'

$$X_\sigma = \tfrac{1}{4}\,\mathrm{Tr}(I+s_1S)(I+t_1T)A_d(u)B_d(u)C_d(u)D_d(u). \tag{3.23}$$

Consistent with the fact that the central charge of the Ashkin-Teller model is $c=1$, we find that the X_σ are products of two $c=\frac{1}{2}$ Virasoro characters

$$X_{(+,+)} = X_{(-,+)} = \chi^1_{1/16}(x^4) = \chi^{1/2}_0(x^4)\chi^{1/2}_{1/16}(x^4) \tag{3.24}$$

$$X_{(+,-)} = X_{(-,-)} = \chi^1_{9/16}(x^4) = \chi^{1/2}_{1/2}(x^4)\chi^{1/2}_{1/16}(x^4). \tag{3.25}$$

Here $x=\exp(-\pi\lambda/\varepsilon)$, $\chi^1_h(q)$ are the $c=1$ chiral characters of the magnetisation operators or twist fields of the associated orbifold conformal field theory (Dijkgraaf *et al* 1989) and $\chi^{1/2}_\Delta(q)$ are the $c=\frac{1}{2}$ Virasoro characters defined by

$$\chi^{1/2}_0(q) = \frac{1}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-1/2}) + \prod_{n=1}^{\infty}(1-q^{n-1/2})\right) \tag{3.26a}$$

$$\chi^{1/2}_{1/2}(q) = \frac{1}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-1/2}) - \prod_{n=1}^{\infty}(1-q^{n-1/2})\right) \tag{3.26b}$$

$$\chi^{1/2}_{1/16}(q) = \prod_{n=1}^{\infty}(1+q^n). \tag{3.26c}$$

The observed scaling dimensions $x=2h$, with $h=0+\frac{1}{16}=\frac{1}{16}$ and $h=\frac{1}{2}+\frac{1}{16}=\frac{9}{16}$ as in (3.24) and (3.25), agree with the magnetic scaling dimensions given in (1.1*b*). A similar calculation, with the centre spin $\sigma=(0,0)$, gives

$$\begin{aligned} X_{(0,0)} &= \mathrm{Tr}\,A_d(u)B_d(u)C_d(u)D_d(u) \\ &= \chi^{1/2}_{1/16}(x^2) \\ &= [\chi^{1/2}_0(x^4)+\chi^{1/2}_{1/2}(x^4)]\chi^{1/2}_{1/16}(x^4). \end{aligned} \tag{3.27}$$

Presumably, we do not see the other characters and scaling dimensions because we can only calculate in the partially ordered regimes.

4. Free energy, scaling and critical behaviour

The Ashkin-Teller model is related by duality (Wegner 1972) to the eight-vertex model. It therefore satisfies (Fendley and Ginsparg 1989) the same inversion relation and has the same free energy

$$\begin{aligned} f &= \lim_{N\to\infty}\frac{1}{N}\log Z \\ &= \log\rho' + \sum_{n=1}^{\infty}\frac{(x^{2n}+q^{2n}x^{-2n})(1-w^{2n})(1-x^{2n}w^{-2n})}{n(1+x^{2n})(1-q^{2n})} \end{aligned} \tag{4.1a}$$

where

$$w=\exp(-\pi u/\varepsilon) \qquad x=\exp(-\pi\lambda/\varepsilon)$$
$$q=\exp(-\pi^2/\varepsilon) \qquad p=\exp(-\varepsilon). \tag{4.1b}$$

This formula applies to the fundamental regime (2.20) and extends to the complete exact solution manifold through the symmetries (2.6). The critical manifold is given by $p=0$. In the spatially isotropic case ($u=\lambda/2$), this manifold lies on the self-dual line

$$\sinh 2K=\exp(-2M) \qquad K=L. \tag{4.2}$$

The crossing parameter λ is related to the interaction M along this critical line by

$$2\cos\lambda=\exp(4M)-1 \tag{4.3}$$

where $0<\lambda<2\pi/3$. The other two critical lines with continuously varying critical behaviour shown in figure 3 are obtained by permuting the interactions K, L and M.

The critical behaviour does not depend on the spatial anisotropy u. As $p\to 0$, the behaviour of the Ashkin-Teller free energy is the same as for the eight-vertex model (Baxter 1982), namely,

$$f\sim p^{2-\bar\alpha} \qquad \bar\alpha=2-\pi/\lambda. \tag{4.4}$$

More generally, scaling theory asserts that the singular part of the free energy is a homogeneous function of the thermodynamic fields

$$\begin{aligned} f_{\text{sing}} &= b^{-2} f_{\text{sing}}(b^{y_t}t,\, b^{y_p}p,\, b^{y_\sigma}h,\, b^{y_e}k,\, u,\, \lambda,\ldots)\\ &= t^{2/y_t}F\left(\frac{p}{t^{y_p/y_t}},\frac{h}{t^{y_\sigma/y_t}},\frac{k}{t^{y_e/y_t}},u,\lambda,\ldots\right)\\ &= p^{2/y_p}G\left(\frac{t}{p^{y_t/y_p}},\frac{h}{p^{y_\sigma/y_p}},\frac{k}{p^{y_e/y_p}},u,\lambda,\ldots\right). \end{aligned} \tag{4.5}$$

Here b is an arbitrary scale factor, t is the leading thermal scaling field, p is the next-to-leading thermal scaling field, h is the scaling field conjugate to the total magnetisation and k is the scaling field conjugate to the polarisation. These scaling forms relate the critical exponents $\bar\alpha$, $\bar\beta$, etc obtained using p as the deviation-from-criticality to the usual critical exponents α, β, etc obtained using t as the deviation-from-criticality variable.

The scaling dimensions x_α are related to the renormalisation exponents y_α by

$$x_\alpha=2-y_\alpha \qquad \alpha=t,p,\sigma,e,\text{etc}. \tag{4.6}$$

The scaling dimensions, as given by (1.1), are

$$x_{m,n}=\frac{m^2}{4(2-y)}+n^2(2-y) \tag{4.7a}$$

$$\begin{aligned} &x_0=0\ (\text{identity}) && x_\sigma=\tfrac18\ (\text{magnetisation})\\ &x'_\sigma=\tfrac98\ (\text{second magnetisation}) && x=2\ (\text{marginal operator}). \end{aligned} \tag{4.7b}$$

From (4.4) and the last scaling form in (4.5) we can read off the free energy exponent

$$2-\bar\alpha=2/y_p=\pi/\lambda. \tag{4.8a}$$

Hence from (3.22), the magnetisation scaling dimension is given by

$$x_\sigma=2\bar\beta_\sigma/(2-\bar\alpha)=(\pi/8\lambda)/(\pi/\lambda)=\tfrac18. \tag{4.8b}$$

This result follows directly from the calculated values of $\bar{\alpha}$ and $\bar{\beta}_\sigma$. We identify p as a next-to-leading thermal scaling field with dimensions

$$y_p = y = 2\lambda/\pi \qquad x_p = 2 - y = x_{0,1}. \tag{4.9}$$

The renormalisation exponent y thus lies in the interval $0 < y < \frac{4}{3}$.

The usual critical exponents (Baxter 1982), defined with t as deviation-from-criticality variable, take the values

$$\begin{aligned} &\alpha = (2-2y)/(3-2y) \qquad &&\beta_\sigma = (2-y)/(24-16y) \\ &\beta_e = 1/(12-8y) \qquad &&\delta_\sigma = 15. \end{aligned} \tag{4.10}$$

This yields

$$x_t = 2 - 2/(2-\alpha) = 1/(2-y) = x_{2,0} \tag{4.11a}$$

$$x_e = 2\Delta_e = 2\beta_e/(2-\alpha) = 1/4(2-y) = x_{1,0} \tag{4.11b}$$

in agreement with (4.7a) and the identification (4.8b).

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